Cronbach's alpha as the mean of all split-half reliabilities

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Reliability

- Reliability of a test or questionnaire
- Correlation between two parallel tests (Lord & Novick 1968)
- Often only one test administration
- Widely used: coefficient alpha (Guttman 1945, Cronbach 1951)
- Likely to be a standard tool in the future

Definition

$$\alpha := \frac{n}{n-1} \cdot \frac{\sum_{i \neq i'} \sigma_{ii'}}{\sigma_X^2}$$

where

- $n \geq 2$ is the number of items
- $\sigma_{ii'}$ the covariance between items i and i'
- σ_X^2 the variance of the total score

Famous description

Cronbach (1951): Mean of all split-half reliabilities

Split-half reliability

- Split tests into two halves
- Correlation between halves is estimate of reliability
- Correct estimate for half test length

Limitations

- Split-half reliability of Flanagan (1937) and Rulon (1937)
- Two halves must have equal size
  number of items must be even

Split-half reliability

Split n into two halves n_1 and n_2 with $n_1 + n_2 = n$

$$p_1 = \frac{n_1}{n} \qquad p_2 = \frac{n_2}{n}$$

Flanagan (1937) and Rulon (1937) proposed

$$\alpha_2 := \frac{4\sigma_{12}}{\sigma_X^2}$$

where

- σ_{12} is the covariance between the sum scores of the two halves

Cronbach (1951): $\alpha = E(\alpha_2)$ if $p_1 = p_2$

Alpha of k-split

$$\alpha_k := \frac{k}{k-1} \cdot \frac{\sum_{j \neq j'} \sigma_{jj'}}{\sigma_X^2}$$

where

- $k = 2, \ldots, n$ is the number of parts
- $\sigma_{jj'}$ the covariance between sumscores of parts j and j'
- σ_X^2 the variance of the total score
- $\alpha_n = \alpha$

Perfect split

Raju (1977, Psychometrika):

- Perfect split: If split is such that k parts have equal size, then alpha is mean of alphas of all possible k-splits
  $\alpha = E(\alpha_k)$
  $\alpha = E(\alpha_2)$ (Cronbach 1951)

Example 12 items

- into (6)(6) (2 parts of size 6)
- into (4)(4)(4)
- into (3)(3)(3)(3)
- into (1)(1)(1)(1)(1)(1)(1)(1)(1)(1)(1)(1)

Research question

Raju (1977, Psychometrika):

- If split is such that parts do not have equal sizes, then alpha exceeds mean of all possible splits
  $\alpha > E(\alpha_k)$

How close are α and $E(\alpha_k)$ in this case?

Formula for $E(\alpha_k)$

Using tools from Raju (1977)

$$E(\alpha_k) = \frac{n}{n-1} \cdot \frac{k}{k-1} \cdot \frac{\sum_{j \neq j'} p_j p_{j'} \sum_{i \neq i'} \sigma_{ii'}}{\sigma_X^2} = \alpha \cdot \frac{k}{k-1} \sum_{j \neq j'} p_j p_{j'}$$

Non-negative difference

$$\alpha - E(\alpha_k) = \alpha \left(1 - \frac{k}{k-1} \sum_{j \neq j'} p_j p_{j'} \right) \leq 1 - \frac{k}{k-1} \sum_{j \neq j'} p_j p_{j'}$$

Difference

α and $E(\alpha_k)$ 'equal' if $\alpha - E(\alpha_k) < 0.01$

Using previous inequality

$$\alpha - E(\alpha_k) \leq 1 - \frac{k}{k-1} \sum_{j \neq j'} p_j p_{j'} < 0.01$$

or

$$\frac{k}{k-1} \sum_{j \neq j'} p_j p_{j'} > 0.99$$

2-split

$2m + 1$ items
$k = 2$ parts
'Best' split is

$$p_1 = \frac{m}{2m+1} \qquad p_2 = \frac{m+1}{2m+1}$$

Check inequality

$$4p_1p_2 > 0.99$$

With ≥ 11 items alpha 'equal' to mean of all split-half reliabilities

m	n	$4p_1p_2$
1	3	0.889
2	5	0.960
3	7	0.980
4	9	0.988
5	**11**	0.992
6	13	0.994
7	15	0.996
8	17	0.997

3-split

$3m + 1$ items
$k = 3$ parts
'Best' split is

$$p_1 = p_2 = \frac{m}{3m+1} \qquad p_3 = \frac{m+1}{3m+1}$$

Check inequality

$$3(p_1 p_2 + p_1 p_3 + p_2 p_3) > 0.99$$

m	n	
1	4	0.938
2	7	0.980
3	**10**	0.990
4	13	0.994
5	16	0.996
6	19	0.997
7	22	0.998
8	25	0.998

Another 3-split

$3m + 2$ items
$k = 3$ parts
'Best' split is

$$p_1 = \frac{m}{3m+2} \qquad p_2 = p_3 = \frac{m+1}{3m+2}$$

Check inequality

$$3(p_1p_2 + p_1p_3 + p_2p_3) > 0.99$$

With ≥ 11 items $\alpha \approx E(\alpha_3)$

m	n	
1	5	0.960
2	8	0.984
3	**11**	0.992
4	14	0.995
5	17	0.997
6	20	0.998
7	23	0.998
8	26	0.999

Worst 2-split

With 'best' splits we have for sufficiently large n

$$\frac{k}{k-1}\sum_{j\neq j'} p_j p_{j'} > 0.99 \qquad \text{and thus} \qquad \alpha - E(\alpha_k) < 0.01$$

Does this hold for any split?

No. Suppose n items and 2-split

$$p_1 = \frac{1}{n} \qquad p_2 = \frac{n-1}{n}$$

Worst possible 2-split. We have

$$4p_1p_2 = \frac{4(n-1)}{n^2} \to 0 \quad \text{as} \quad n \to \infty$$

References

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