International Mathematical Forum, Vol. 12, 2017, no. 18, 861 - 868
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2017.7874

If d is Super-Metric, Then $d/(1+d)$ is Super-Metric

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Abstract

If a function d is metric, a well-known result is that $d/(1+d)$ is also metric. We consider m-ary analogs of the binary notion of semi-metric, called hemi-metrics and super-metrics. The metrics are totally symmetric maps from X^{m+1} into $\mathbb{R}_{\geq 0}$. It is shown that, if d is super-metric, then $d/(1+d)$ is also super-metric.

Mathematics Subject Classification: 51Fxx, 54E35

Keywords: Hemi-metric, simplex inequality, tetrahedron inequality

1 Hemi-metrics and super-metrics

A metric is a function that defines a distance between two elements of a set. We consider generalizations of the notion of metric in the direction of distances between three or more elements.

Deza and Rosenberg [4] introduced the following notion. Let m be a positive integer and X a set with at least $m+2$ elements. A function $d : X^{m+1} \to \mathbb{R}$ is called *m-hemi-metric* if (see, also [1,2,5]):

1. d is *non-negative*, i.e., $d(x_1, \ldots, x_{m+1}) \geq 0$ for all $x_1, \ldots, x_{m+1} \in X$.

2. d is *totally symmetric*, i.e., satisfies $d(x_1, \ldots, x_{m+1}) = d(x_{\pi(1)}, \ldots, x_{\pi(m+1)})$ for all $x_1, \ldots, x_{m+1} \in X$ and for any permutation π of $\{1, \ldots, m+1\}$.

3. d is *zero conditioned*, i.e. $d(x_1, \ldots, x_{m+1}) = 0$ if and only if $x_1, \ldots, x_{m+1}$ are not pairwise distinct.

4. For all $x_1, \ldots, x_{m+2} \in X$, d satisfies the *m-simplex inequality*:

$$d(x_1, \ldots, x_{m+1}) \leq \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}). \tag{1}$$

The notion of m-hemi-metric is an m-ary analog of the binary notion of semi-metric. An important special case of the m-hemi-metric is the following notion obtained for $m = 2$. A function $d : X^3 \to \mathbb{R}$ is called a *2-metric* if d is non-negative, totally symmetric, zero conditioned, and satisfies the *tetrahedron inequality*:

$$d(x_1, x_2, x_3) \leq d(x_1, x_2, x_4) + d(x_1, x_3, x_4) + d(x_2, x_3, x_4). \tag{2}$$

Interpreting $d(x_1, x_2, x_3)$ as the area of the triangle with vertices x_1, x_2 and x_3, the tetrahedron inequality specifies that the area of each triangle face of the tetrahedron formed by x_1, x_2, x_3 and x_4 does not exceed the sum of the areas of the remaining faces. Alternative axiom systems are considered in [6-11].

Deza and Dutour [3] introduced the following notion. Let s be a positive real number. A function $d : X^{m+1} \to \mathbb{R}$ is called *(m, s)-super-metric* if d is non-negative, totally symmetric, zero conditioned, and satisfies the *(m, s)-simplex inequality*

$$sd(x_1, \ldots, x_{m+1}) \leq \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}). \tag{3}$$

An (m, s)-super-metric is an m-hemi-metric if $s \geq 1$. Furthermore, a m-hemi-metric is a $(m, 1)$-super-metric and a semi-metric is a $(1, 1)$-super-metric.

For the ordinary metric, a well-known result is that, if d is metric, then $d/(1 + d)$ and $\min\{1, d\}$ are also metric. In Section 2 we present an analogous result for the function $d/(1+d)$ for hemi-metrics and super-metrics. In Section 3 we present an analogous result for the function $\min\{1, d\}$ for hemi-metrics and the $(2, 2)$-super-metric.

2 Function $d/(1 + d)$

Lemma 2.1 considers the notion of m-hemi-metric. Lemma 2.3 considers the notion of (m, s)-super-metric for $s \geq 1$. Lemma 2.2 is used in the proof of Lemmas 2.3 and 3.2.

Lemma 2.1. *Let d be m-hemi-metric. Then $d/(1 + d)$ is m-hemi-metric.*

Proof. Non-negativity of $d/(1+d)$ follows from the non-negativity of d. Furthermore, total symmetry and axiom 3 follow from the identity

$$\frac{d(x_1,\dots,x_{m+1})}{1+d(x_1,\dots,x_{m+1})} = 1 - \frac{1}{1+d(x_1,\dots,x_{m+1})}, \tag{4}$$

and the fact that d is totally symmetric and zero conditioned. Thus, we must show that $d/(1+d)$ satisfies (1).

Because $d/(1+d)$ is strictly increasing in d, and since d satisfies (1), we have

$$\begin{aligned}\frac{d(x_1,\dots,x_{m+1})}{1+d(x_1,\dots,x_{m+1})} &\le \frac{\sum_{i=1}^{m+1} d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2})}{1+\sum_{i=1}^{m+1} d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2})} \\ &= \sum_{i=1}^{m} \frac{d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2})}{1+\sum_{j=1}^{m+1} d(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{m+2})}. \end{aligned} \tag{5}$$

Furthermore, for all $i \in \{1,\dots,m+1\}$ we have the inequality

$$\begin{aligned}&\frac{d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2})}{1+\sum_{j=1}^{m+1} d(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{m+2})} \\ &\qquad \le \frac{d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2})}{1+d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2})}. \end{aligned} \tag{6}$$

Summing (6) over all $i \in \{1,\dots,m+1\}$, and combining the resulting inequality with inequality (5), completes the proof. □

Lemma 2.2. *Suppose $s > 1$ and let d be (m,s)-super-metric. Then d satisfies the inequality*

$$(s-1)d(x_1,\dots,x_{m+1}) \le \sum_{i=2}^{m+1} d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2}). \tag{7}$$

Proof. Interchanging the roles of x_1 and x_{m+2} in (3), and dividing the result by s, we obtain

$$d(x_2,\dots,x_{m+2}) \le \frac{1}{s}\sum_{i=2}^{m+2} d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2}). \tag{8}$$

Adding inequalities (3) and (8) yields

$$\left(s-\frac{1}{s}\right) d(x_1,\dots,x_{m+1}) \le \left(1+\frac{1}{s}\right)\sum_{i=2}^{m+1} d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2}), \tag{9}$$

which is equivalent to (7). □

Lemma 2.3. *Suppose $s \geq 1$ and let d be (m,s)-super-metric. Then $d/(1+d)$ is (m,s)-super-metric.*

Proof. The case $s = 1$ is proved in Lemma 2.1. Therefore, suppose $s > 1$. The proof of non-negativity, total symmetry and axiom 3 is analogous to the proof of Lemma 2.1. We must show that d satisfies (3).

Because $d/(1+d)$ is strictly increasing in d, and since d satisfies (3), we have

$$\frac{d(x_1,\ldots,x_{m+1})}{1+d(x_1,\ldots,x_{m+1})} \leq \frac{\frac{1}{s}\sum\limits_{i=1}^{m+1} d(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+2})}{1+\frac{1}{s}\sum\limits_{i=1}^{m+1} d(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+2})}. \tag{10}$$

After multiplying both sides of (10) by s, we may write the result as

$$\frac{sd(x_1,\ldots,x_{m+1})}{1+d(x_1,\ldots,x_{m+1})} \leq \sum_{i=1}^{m+1} \frac{d(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+2})}{1+\frac{1}{s}\sum\limits_{j=1}^{m+1} d(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{m+2})}. \tag{11}$$

Due to Lemma 2.2, combined with the total symmetry of d, we have for all $i \in \{1,\ldots,m+1\}$,

$$\begin{aligned}(s-1)d(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+2}) \leq \sum_{j=1}^{m+1} & d(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{m+2}) \\ & - d(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+2}).\end{aligned} \tag{12}$$

Adding $d(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+2})$ to both sides of (12), and dividing the result by s, we have for all $i \in \{1,\ldots,m+1\}$,

$$d(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+2}) \leq \frac{1}{s}\sum_{j=1}^{m+1} d(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{m+2}). \tag{13}$$

Furthermore, using (13), we have, for all $i \in \{1,\ldots,m+1\}$, the inequality

$$\begin{aligned}&\frac{d(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+2})}{1+\frac{1}{s}\sum\limits_{j=1}^{m+1} d(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{m+2})} \\ &\qquad \leq \frac{d(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+2})}{1+d(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+2})}.\end{aligned} \tag{14}$$

Summing (14) over all $i \in \{1,\ldots,m+1\}$, and combining the result with (11), completes the proof. □

3 Function $\min\{1,d\}$

Lemma 3.1 considers the notion of m-hemi-metric. Lemma 3.2 considers the notion of $(2,2)$-super-metric.

Lemma 3.1. *Let d be m-hemi-metric. Then $\min\{1,d\}$ is m-hemi-metric.*

Proof. Non-negativity, symmetry and axiom 3 of $\min\{1,d\}$ follow from the analogous properties of d. Thus, we must show that $\min\{1,d\}$ satisfies (1). We go through the various cases.

Suppose there is an $j \in \{1,\dots,m+1\}$ such that

$$d(x_1,\dots,x_{m+1}) \le d(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{m+2}). \tag{15}$$

In this case we have

$$\begin{aligned}\min\{1,d(x_1,\dots,x_{m+1})\} &\le \min\{1,d(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{m+2})\}\\ &\le \sum_{i=1}^{m+1}\min\{1,d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2})\}.\end{aligned} \tag{16}$$

Thus, we may assume that, for all $i \in \{1,\dots,m+1\}$, we have

$$d(x_1,\dots,x_{m+1}) \ge d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2}). \tag{17}$$

Suppose $d(x_1,\dots,x_{m+1}) \le 1$. In this case we have, for all $i \in \{1,\dots,m+2\}$,

$$\min\{1,d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2})\} = d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2}),$$

and it follows that $\min\{1,d\}$ satisfies (1) because d satisfies (1).

Next, suppose $d(x_1,\dots,x_{m+1}) > 1$. Furthermore, suppose there is an $j \in \{1,\dots,m+1\}$ such that $d(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{m+2}) \ge 1$. In this case we have

$$\begin{aligned}\min\{1,d(x_1,\dots,x_{m+1})\} = 1 &= \min\{1,d(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{m+2})\}\\ &\le \sum_{i=1}^{m+1}\min\{1,d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2})\}.\end{aligned} \tag{18}$$

Therefore, suppose that $d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2}) \le 1$ for all $i \in \{1,\dots,m+1\}$. In this final case we have, since d satisfies (1),

$$\begin{aligned}\min\{1,d(x_1,\dots,x_{m+1})\} = 1 &< d(x_1,\dots,x_{m+1})\\ &\le \sum_{i=1}^{m+1} d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2})\\ &= \sum_{i=1}^{m+1}\min\{1,d(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{m+2})\}.\end{aligned} \tag{19}$$

This completes the proof. □

Lemma 3.2. *Let d be $(2,2)$-super-metric. Then* $\min\{1,d\}$ *is $(2,2)$-super-metric.*

Proof. Non-negativity, symmetry and axiom 3 of $\min\{1,d\}$ follow from the analogous properties of d. Thus, we must show that $\min\{1,d\}$ satisfies

$$2d(x_1,x_2,x_3) \leq d(x_1,x_2,x_4) + d(x_1,x_3,x_4) + d(x_2,x_3,x_4), \qquad (20)$$

which is a strong version of tetrahedron inequality (2) [6,8,9,11]. We go through the various cases.

First, suppose $d(x_1,x_2,x_3) \leq 1$. In addition, suppose at least two of the three quantities on the right-hand side of (20) ≥ 1. In this case we have

$$\begin{aligned} 2\min\{1,d(x_1,x_2,x_3)\} &= 2d(x_1,x_2,x_3) \leq 2 = 1+1 \\ &\leq \min\{1,d(x_1,x_2,x_4)\} + \min\{1,d(x_1,x_3,x_4)\} + \min\{1,d(x_2,x_3,x_4)\}. \end{aligned}$$

Furthermore, without loss of generality, suppose that $d(x_1,x_2,x_4) > 1$ and $d(x_1,x_3,x_4), d(x_2,x_3,x_4) \leq 1$. In this case we have

$$\min\{1,d(x_1,x_2,x_3)\} = d(x_1,x_2,x_3) \leq 1 = \min\{1,d(x_1,x_2,x_4)\}. \qquad (21)$$

We also have, using Lemma 2.2,

$$\begin{aligned} \min\{1,d(x_1,x_2,x_3)\} &= d(x_1,x_2,x_3) \leq d(x_1,x_3,x_4) + d(x_2,x_3,x_4) \\ &= \min\{1,d(x_1,x_3,x_4)\} + \min\{1,d(x_2,x_3,x_4)\}. \end{aligned} \qquad (22)$$

Combining (21) and (22) gives the desired inequality.

Moreover, suppose all three quantities on the right-hand side of (20) ≤ 1. In this case we have, since d satisfies (20),

$$\begin{aligned} 2\min\{1,d(x_1,x_2,x_3)\} &= 2d(x_1,x_2,x_3) \\ &\leq d(x_1,x_2,x_4) + d(x_1,x_3,x_4) + d(x_2,x_3,x_4) \\ &= \min\{1,d(x_1,x_2,x_4)\} + \min\{1,d(x_1,x_3,x_4)\} + \min\{1,d(x_2,x_3,x_4)\}. \end{aligned}$$

Second, suppose $d(x_1,x_2,x_3) > 1$. In addition, suppose at least two of the three quantities on the right-hand side of (20) ≥ 1. In this case we have

$$\begin{aligned} 2\min\{1,d(x_1,x_2,x_3)\} &= 2 = 1+1 \\ &\leq \min\{1,d(x_1,x_2,x_4)\} + \min\{1,d(x_1,x_3,x_4)\} + \min\{1,d(x_2,x_3,x_4)\}. \end{aligned}$$

Furthermore, without loss of generality, suppose that $d(x_1,x_2,x_4) \geq 1$ and $d(x_1,x_3,x_4), d(x_2,x_3,x_4) \leq 1$. In this case we have

$$2\min\{1,d(x_1,x_2,x_3)\} = 2 < d(x_1,x_2,x_3) + \min\{1,d(x_1,x_2,x_4)\}. \qquad (23)$$

We also have, using Lemma 2.2,

$$\begin{aligned} d(x_1, x_2, x_3) &\leq d(x_1, x_3, x_4) + d(x_2, x_3, x_4) \\ &= \min\{1, d(x_1, x_3, x_4)\} + \min\{1, d(x_2, x_3, x_4)\}. \end{aligned} \quad (24)$$

Combining (23) and (24) gives the desired inequality.

Finally, suppose all three quantities on the right-hand side of (20) ≤ 1. In this case we have, since d satisfies (20),

$$\begin{aligned} 2\min\{1, d(x_1, x_2, x_3)\} &= 2 < 2d(x_1, x_2, x_3) \\ &\leq d(x_1, x_2, x_4) + d(x_1, x_3, x_4) + d(x_2, x_3, x_4) \\ &= \min\{1, d(x_1, x_2, x_4)\} + \min\{1, d(x_1, x_3, x_4)\} + \min\{1, d(x_2, x_3, x_4)\}. \end{aligned}$$

This completes the proof. □

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Received: September 5, 2017; Published: October 4, 2017