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Short communication

The Cicchetti–Allison weighting matrix is positive definite

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ABSTRACT

Application of Cohen's weighted kappa for inter-rater agreement requires the specification of a weighting matrix. An explicit formula for the determinants of the principal minors of the weighting matrix with Cicchetti–Allison weights is derived. Since all determinants are strictly positive, it follows that the Cicchetti–Allison weighting matrix is positive definite.

1. Weighted kappa

Analysis of agreement between two raters is often used to assess the reliability of a categorical rating system. The raters may be pathologists that rate the severity of lesions from scans (Crewson, 2005; Kundel and Polansky, 2003) or competing diagnostic devices that classify the extent of disease in patients into ordinal categories (Warrens, 2012). High agreement between the ratings indicates consensus in the diagnosis and interchangeability of the measuring devices. Standard tools for assessing agreement are the descriptive statistics unweighted kappa for ratings on a nominal scale (Cohen, 1960; Yang and Chinchilli, 2009; Warrens, 2010) and weighted kappa for ratings on an ordinal scale (Cohen, 1968; Warrens, 2013).

Weighted kappa allows the assignment of weights to describe the closeness of agreement between the $n \geq 2$ categories. Let $w_{ij} \in [0, 1]$ denote the agreement weight between categories i and j ($i, j \in \{1, 2, \ldots, n\}$). The equality $w_{ij} = 1$ reflects that there is perfect agreement when a target is assigned to category i by rater 1 and category j by rater 2, whereas $0 < w_{jk} < 1$ reflects some disagreement when a target is assigned to different categories by the raters. The most popular weighting matrix (Maclure and Willett, 1987; Graham and Jackson, 1993) is the matrix with quadratic weights (Fleiss and Cohen, 1973; Warrens, 2012). Another popular weighting matrix is the matrix with linear weights (Cicchetti and Allison, 1971; Vanbelle and Albert, 2009; Warrens, 2011) given by

$$W_n = \left[1 - \frac{|i-j|}{n-1}\right]_{i,j\in\{1,2,\ldots,n\}} = \begin{bmatrix} 1 & \frac{n-2}{n-1} & \cdots & \frac{1}{n-1} & 0 \\ \frac{n-2}{n-1} & 1 & \cdots & \frac{2}{n-1} & \frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{2}{n-1} & \cdots & 1 & \frac{n-2}{n-1} \\ 0 & \frac{1}{n-1} & \cdots & \frac{n-2}{n-1} & 1 \end{bmatrix}.$$

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Yang and Chinchilli (2011, p. 1064) conjecture that the $n \times n$ matrix W_n with Cicchetti–Allison weights is non-negative definite. The non-negative definite nature of a weighting matrix is a sufficient condition for the existence of a representation of the categories in Euclidean space (Gower, 1966). The weighting matrix can thus be subjected to various multivariate data analysis techniques. In the next section we show that Yang and Chinchilli's conjecture is correct by proving that the matrix W_n is positive definite.

2. Cicchetti–Allison weights

Fix $n \geq 2$, let $m \in \{1, 2, \ldots, n\}$, and consider the $m \times m$ matrix

$$U_m^{(n)} = \begin{bmatrix} n-1 & n-2 & \cdots & n-m+1 & n-m \\ n-2 & n-1 & \cdots & n-m+2 & n-m+1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n-m+1 & n-m+2 & \cdots & n-1 & n-2 \\ n-m & n-m+1 & \cdots & n-2 & n-1 \end{bmatrix}.$$

We have the following expression for the determinant of $U_m^{(n)}$.

Lemma. $\det U_m^{(n)} = (2n-m-1)2^{m-2}$.

Proof. We use the property that adding a scalar multiple of one row (column) to another row (column) does not change the value of the determinant (see, e.g. Bloom, 1979, p. 306). If we keep the first row unaltered and replace for $i \geq 2$ the i-th row by the difference between the $(i-1)$-th and i-th row, we obtain

$$\det U_m^{(n)} = \begin{vmatrix} n-1 & n-2 & n-3 & \cdots & n-m \\ 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & -1 & \cdots & -1 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -1 \end{vmatrix}.$$

Next, if we keep the first column unaltered and replace for $i \geq 2$ the i-th column by the difference between the $(i-1)$-th and i-th column, we obtain

$$\det U_m^{(n)} = \begin{vmatrix} n-1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 0 & 2 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 2 \end{vmatrix}.$$

Finally, we have

$$\det U_m^{(n)} = \frac{1}{2}\begin{vmatrix} 2(n-1) & 1 & 1 & \cdots & 1 \\ 2 & 2 & 0 & \cdots & 0 \\ 2 & 0 & 2 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 2 & 0 & 0 & \cdots & 2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2n-m-1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 2 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{vmatrix},$$

from which the formula follows. □

The $m \times m$ matrix $W_m^{(n)} = (n-1)^{-1}U_m^{(n)}$ is the m-th principal minor of W_n, where $W_n^{(n)} = W_n$. It follows from the lemma that the determinant

$$\det W_m^{(n)} = \det \frac{U_m^{(n)}}{n-1} = \frac{(2n-m-1)2^{m-2}}{(n-1)^m}.$$

Since $2n > m+1$ for $n \geq 2$ and $1 \leq m \leq n$ we have $\det W_m^{(n)} > 0$ for $n \geq 2$ and $1 \leq m \leq n$. Since all its principal minors have strictly positive determinants, it follows that W_n is positive definite (see, e.g. Bloom, 1979, p. 547).

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